

Effective QCD string beyond the Nambu–Goto action

J. Ambjørn ^{a,b}, Y. Makeenko ^{a,c} and A. Sedrakyan ^{a,d}

^a *The Niels Bohr Institute, Copenhagen University,
Blegdamsvej 17, DK-2100 Copenhagen, Denmark*

^b *IMAPP, Radboud University, Heyendaalseweg 135, 6525 AJ, Nijmegen, The Netherlands*

^c *Institute of Theoretical and Experimental Physics,
B. Chermushkinskaya 25, 117218 Moscow, Russia*

^d *Yerevan Physics Institute, Br. Alikhanyan str 2, Yerevan-36, Armenia
email: ambjorn@nbi.dk makeenko@nbi.dk sedrak@nbi.dk*

We consider the QCD string as an effective string, whose action describes long-range stringy fluctuations. The leading infrared contribution to the ground state energy is given by the Alvarez–Arvis formula, usually derived using the Nambu–Goto action. Here we rederive it by a saddle point calculation using the Polyakov formulation of the free string, where the world sheet metric and the target space coordinates are treated as independent variables. The next-order relevant in the infrared term in the effective action is the extrinsic curvature term. We show that the spectrum does not change order by order in the inverse string length, but may change at intermediate distances.

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I. INTRODUCTION

As is well understood by now, QCD string is made out of fluxes of the gluon field. Effective stringy degrees of freedom make sense at the distances larger than the confinement scale – at shorter distances quark-gluon degrees of freedom are more appropriate owing to asymptotic freedom. The corresponding effective theory of long strings can be constructed [1] order by order in the inverse string length and can be consistently quantized in any number of space-time dimensions. Various approaches to such an effective string theory have recently attracted considerable attention [2–24].

The analytical results concern mostly the simplest bosonic string given by the Nambu–Goto area action or the Polyakov action. However, such an action can be only serve as an approximation to the full effective action for the QCD string at very large distances. With decreasing the distance terms less dominant in the infrared will become essential and are needed in order to avoid in particular the tachyonic instability of the Nambu–Goto energy spectrum.

The investigation of the effective string theory is therefore twofold: firstly, to quantize the pure bosonic Nambu–Goto long string and to compute its spectrum and, secondly, to construct the next relevant operators and to compute their contribution to the spectrum. We shall deal in this Paper with both of these tasks.

In Sect. II we concentrate on the spectrum of the Nambu–Goto effective string, using its formulation *à la* Polyakov [25]. We fix the conformal gauge $g_{ab} = e^\phi \delta_{ab}$ and treat the target-space string coordinates and the world sheet metric as independent. In a Wilson loop setup they couple via the boundary conditions and we will show how one can reproduce the Alvarez–Arvis stringy spectrum [26, 27] by a saddle point calculation.

The essential feature of the Liouville action¹

$$S_g = -\frac{d-26}{96\pi} \int d^2z \left[(\partial_a \phi)^2 + \mu^2 e^\phi \right] \quad (1)$$

is the fact that it depends only on the metric, i.e. internal geometry of the surface, and does not depend on external geometry of its embedding into d -dimensional space. This is a quite strong restriction on string properties, which is dictated by the original bare string action [25]. However, for other strings, for example for the Green–Schwarz superstring, where the fermions belong to the spinor representation of the group $SO(d)$ of the target space, the effective action of strings depends essentially on external geometry of the string world sheet [28–30] due to quantum fluctuations of fermions. Namely, the effective action depends now on the second quadratic form h_{ab} of the embedded surface. The simplest model with an inclusion of external geometry – the extrinsic curvature – was considered in [31, 32]. In general, one can expect that similar dynamics of external geometry of embedded surfaces may appear also in an effective theory of QCD string.

In Sect. III we consider the bosonic string with extrinsic curvature. We briefly review the standard results [33–37] for the ground-state energy in the world sheet parametrization and emphasize that deviations from the spectrum of the Nambu–Goto string are explicitly seen, when the coefficient in front of the extrinsic-curvature term in the action becomes large. The string then becomes rigid. We reproduce the same results in the language of the effective string, using the upper half-plane

¹ Here d is the central charge of matter fields living on a string world sheet. For bosonic string it is just the dimension of space-time. For fermions it counts the number of Dirac particles $2^{[d/2]}$, each of which has the central charge 1.

parametrization. This now happens because the conformal anomaly changes. Finally, in Sect. IV we derive the most general expression for the conformal anomaly, accounting for its possible dependence on external geometry, and discuss the effect of these additional terms on the spectrum.

II. THE EFFECTIVE NAMBU-GOTO STRING

A. Effective string: Polyakov's formulation

In the Polyakov string formulation the coordinates $X_\mu(x, y)$ of string in the target space and the metric $g_{ab}(x, y)$ of the world sheet spanned by the string are independent. The Polyakov path integral is thus [25]:

$$W(C) = \int \mathcal{D}g_{ab} \int \mathcal{D}X_\mu e^{-\frac{1}{4\pi\alpha'} \int d^2z \sqrt{g} g^{ab} \partial_a X_\mu \cdot \partial_b X_\mu}, \quad (2)$$

where C denotes a planar curve in R^d (the Wilson loop in the QCD terminology). It is important to recall that the (regularized) functional integration measure $\mathcal{D}X_\mu$ depends on the metric g_{ab} , as the cutoff does for reparametrization invariance. This fact can be viewed as the origin of the conformal anomaly of the theory.

It is convenient to make a partial gauge fixing of $g_{ab}(x, y)$ to the conformal gauge $g_{ab} = e^\phi \delta_{ab}$, where the scalar (Ricci) curvature $R = -\Delta\phi = -e^{-\phi} \partial^2 \phi$. This can be done globally when the topology of the world sheet is that of a disk, which is the situation we consider here. Thus the setup is the following: we have a region D in the complex plane with the topology of the disk. It has a boundary ∂D . The coordinates $X_\mu(z)$, $\mu = 1, \dots, d$ represent a map from D to R^d . The Wilson loop C is a $R \times T$ rectangle in R^d . We assume it is located in the (1,2) plane of R^d . $X_\mu(z)$ maps the boundary ∂D onto C .

Let C be given by the parametrization $x_\mu(s)$, s denoting the arc-length from a point $x_\mu(s_0)$, and assume that $z_0 \in \partial D$ is mapped to $x_\mu(s_0)$. As is described in [38, 39], the natural coupling between the length scale of the Wilson loop C in R^d and the length scale set by the intrinsic metric at the boundary takes place at the boundary by insisting that $\dot{X}_\mu^2(z) \propto e^{\phi(z)}$, $z \in \partial D$. Here $\dot{X}_\mu(z)$ denotes $dX_\mu(z)/dz$ along the boundary. Denote the length of ∂D calculated using the boundary metric $e(z) = e^{\phi(z)/2}$ by L_ϕ and the length of the Wilson loop in R^d (calculated using $X_\mu(z)$) by L ($L = 2(T + R)$). Then we have for a given $\phi(z)$ the boundary condition for $X_\mu(z)$:

$$X_\mu(z) = x_\mu(s(\phi; z)), \quad s(\phi; z) = \frac{L}{L_\phi} \int_{z_0}^z dz e^{\phi(z)/2} \quad (3)$$

for $z \in \partial D$. Without loss of generality we can choose the constant of proportionality between $\dot{X}_\mu^2(z)$ and $e^{\phi(z)}$ to be one, i.e. $L_\phi = L$. Let us introduce the notation $\psi(z) = \phi(z)$ for $z \in \partial D$. $\psi(z)$ is a field which lives on

the boundary ∂D and a change of $\psi(z)$ corresponds to a reparametrization of the boundary.

For a given choice of $\psi(z)$ the equations of motion for $X_\mu(z)$ with corresponding boundary equations become

$$\begin{aligned} \partial^2 X_\mu(z) &= 0, & z \in D, \\ X_\mu(z) &= x_\mu(s(\psi; z)), & z \in \partial D. \end{aligned} \quad (4)$$

Denote the corresponding solution by $X_\mu^\psi(z)$. Similarly, the equations of motion for ϕ leads to the condition that the induced metric

$$(g_{cl})_{ab} \equiv \partial_a X^\psi \cdot \partial_b X^\psi \quad (5)$$

is conformal (or isothermal)

$$(g_{cl})_{ab} = \frac{1}{2} \delta_{ab} \sum_c (g_{cl})_{cc}. \quad (6)$$

If Eqs. (4) and (6) are satisfied, the surface X_μ is the surface of minimal area corresponding to the curve C .

We can now decompose $X_\mu(z)$ as

$$X_\mu(z) = X_\mu^\psi(z) + \delta X_\mu(z), \quad \delta X_\mu(z) = 0 \quad z \in \partial D. \quad (7)$$

The path integral over $X_\mu(z)$ in (2) can be performed by shifting the integral to $\delta X_\mu(z)$ and is given by the conformal anomaly [25]. The end result² is [38]:

$$\begin{aligned} W(C) &= \int \mathcal{D}\psi(z) e^{-\frac{1}{4\pi\alpha'} \int d^2z \partial_a X^\psi \cdot \partial_a X^\psi} \\ &\times \int \mathcal{D}\phi(z) e^{\frac{d-26}{96\pi} \int d^2z \partial_a \phi \partial_a \phi}, \end{aligned} \quad (8)$$

where the functional integral over the Liouville field $\phi(z)$, $z \in D$, is performed with the boundary condition $\phi(z) = \psi(z)$, $z \in \partial D$. It is important to remember that $\mathcal{D}\phi(z)$ is not the measure of a free field $\phi(z)$ since a reparametrization-invariant cutoff is needed in the functional integral, and this cutoff will then depend on ϕ itself, since ϕ is part of the metric.

The Liouville action with the factor d appears as a result of the integration over the δX_μ and reflects that the measure $\mathcal{D}(\delta X_\mu)$ is not invariant under conformal transformations of the intrinsic metric $g_{ab}(z) = e^{\phi(z)} \delta_{ab}$. The factor -26 appears as a result of the partial gauge fixing of $\mathcal{D}g_{ab}$ to conformal gauge, leaving only the conformal mode $\phi(z)$ to be integrated over, the $\phi(z)$ appearing in the path integral in (8) with boundary condition $\phi(z) = \psi(z)$, $z \in \partial D$.

It is custom to represent the disk amplitude (8) as

$$W(C) = e^{-S_{\text{eff}}(C)}, \quad (9)$$

² We have not written here the boundary terms found in [39, 40]. They will be presented below (see Eq. (B5)).

introducing an effective action $S_{\text{eff}}(C)$. It is called effective since quantum fluctuations of X_μ are already taken into account by the presence of the conformal anomaly. As we shall see below, the upper half-plane parametrization is most convenient at this point.

The field ϕ (often called the Liouville field) is generically quantum, but for long strings and/or large $|d|$ it freezes near the value, minimizing an effective action. We then have

$$\langle g_{ab} \rangle_\phi = \rho \delta_{ab}, \quad (10)$$

where ρ coincides with the classical induced Weyl factor ρ_{cl} only when $\alpha' \rightarrow 0$. The explicit formulas for ρ and ρ_{cl} will be shortly presented.

The difference from the Polchinski–Strominger effective string theory [1], where g_{ab} equals the induced metric at the outset, is that we now determine $\langle g_{ab} \rangle_\phi$ by the minimization.

B. UHP parametrization

Since our Wilson loop C has been chosen as a $R \times T$ rectangle (which we can view as lying in the complex w -plane), an obvious solution to Eq. (4) is to choose D as a $\omega_R \times \omega_T$ rectangle, and

$$X_1^\psi = \frac{R}{\omega_R} \Re \omega, \quad X_2^\psi = \frac{T}{\omega_T} \Im \omega, \quad (11)$$

while the rest of the X_μ 's equal 0. This is called the world sheet parametrization. However, we are interested in using instead the upper half-plane (UHP). The reason is that we want to perform the boundary path integral in (8) and this is best done using the upper half-plane parametrization.

For the upper half-plane parametrization $z = x + iy$ with $y \geq 0$ and the boundary is at the real axis ($y = 0$) to be parametrized by s . The Schwarz–Christoffel map of the upper half-plane onto a rectangle is

$$\begin{aligned} \omega(z) &= \sqrt{s_{42}s_{31}} \int_{s_2}^z \frac{dx}{\sqrt{(s_4 - x)(s_3 - x)(x - s_2)(x - s_1)}} \\ &= 2F \left(\sqrt{\frac{s_{31}(z - s_2)}{s_{32}(z - s_1)}}, \sqrt{\frac{s_{32}s_{41}}{s_{42}s_{31}}} \right), \end{aligned} \quad (12)$$

where $s_1 < s_2 < s_3 < s_4$ ($s_{ij} = s_i - s_j$) are associated with the four corners of the rectangle. In Eq. (12) F is the incomplete elliptic integral of the first kind and the normalization factor is introduced for latter convenience.

The new variable ω takes values inside a $\omega_R \times \omega_T$ rectangle, which now can be given the meaning of a world sheet parametrization. From Eq. (12) we have

$$\omega_R = 2K(\sqrt{1-r}), \quad \omega_T = 2K(\sqrt{r}), \quad (13)$$

where K is the complete elliptic integral of the first kind and

$$r = \frac{s_{43}s_{21}}{s_{42}s_{31}} \quad (14)$$

is the projective-invariant ratio.

The classical string configuration is now given by Eq. (11) with $\omega(z)$ given by Eq. (12) and ω_T, ω_R given by Eq. (13). The classical induced metric (5) is diagonal and becomes conformal if

$$\frac{\omega_T}{\omega_R} \equiv \frac{K(\sqrt{r})}{K(\sqrt{1-r})} = \frac{T}{R}, \quad (15)$$

which guarantees that the quadratic term equals the Nambu–Goto one. The ratio of the K 's in Eq. (15) is known as the Grötzsch modulus which is monotonic in r .

For the solution (11) we have for the (diagonal) induced metric

$$\sqrt{g_{\text{cl}}(x, y)} = \frac{RT}{\omega_R \omega_T} \frac{s_{42}s_{31}}{\prod_{i=1}^4 \sqrt{(x - s_i)^2 + y^2}}, \quad (16)$$

which becomes singular at the boundary for $x = s_i$. We thus regularize by moving the boundary from the real axis slightly into the complex plane, i.e. replacing $y = 0$ by $y = \epsilon(s)$ and denoting $\epsilon_i = \epsilon(s_i)$. The boundary metric

$$e(s) \equiv \sqrt[4]{g(s, \epsilon(s))} \quad (17)$$

has then gotten regularized:

$$e(s_j) = \sqrt{\frac{RT}{\omega_R \omega_T}} \frac{s_{42}s_{31}}{\prod_{i \neq j} \sqrt{|s_j - s_i| \epsilon_j}}. \quad (18)$$

On the other hand

$$\epsilon(s) = \epsilon/e(s) \quad (19)$$

and

$$\epsilon_i = \epsilon/e(s_i), \quad (20)$$

where ϵ is a physical cutoff of the dimension of length, for general covariance, as was pointed out in [25]. We then obtain

$$\begin{aligned} \epsilon_1 &= \epsilon^2 \frac{\omega_R \omega_T}{RT} \frac{s_{41}s_{21}}{s_{42}}, \\ \epsilon_2 &= \epsilon^2 \frac{\omega_R \omega_T}{RT} \frac{s_{32}s_{21}}{s_{31}}, \\ \epsilon_3 &= \epsilon^2 \frac{\omega_R \omega_T}{RT} \frac{s_{43}s_{32}}{s_{42}}, \\ \epsilon_4 &= \epsilon^2 \frac{\omega_R \omega_T}{RT} \frac{s_{43}s_{41}}{s_{31}}. \end{aligned} \quad (21)$$

While Eq. (19) is quite general, it was understood in [38] that for a rectangle it means a regularization of the geodesic curvature

$$k_g(s) = -\frac{1}{2e(s)} \partial_y \log \sqrt{g} \Big|_{y=\epsilon/e(s)} \quad (22)$$

at the corners. For thus smeared rectangle we have

$$k_g(s_i) = \frac{1}{2\epsilon}, \quad (23)$$

regularizing divergences at the corners in a nice way.

C. Boundary action

If a function is harmonic in a simple connected domain D , it is uniquely determined by its values on the boundary ∂D by the general Poisson formula. This formula comes simple and useful if we map D to the upper half-plane such that we have a harmonic function $f(x, y)$ given by its value $f(s, 0)$ at the boundary. We now apply this to a solution $X_\mu^\psi(z)$ of (4) and thus rewrite the first term on the right-hand side of Eq. (8) as a boundary contribution

$$\begin{aligned} & \frac{1}{2} \int d^2 z \partial_a X^\psi \cdot \partial_a X^\psi \\ &= \frac{1}{4\pi} \int_{-\infty}^{+\infty} ds_1 ds_2 \frac{[X^\psi(s_1, 0) - X^\psi(s_2, 0)]^2}{(s_1 - s_2)^2}. \end{aligned} \quad (24)$$

The boundary curve C is generically described by the function $x^\mu(t)$ with a certain choice of the parameter t . If we change the parametrization $t \rightarrow f(t)$ ($f'(t) \geq 0$), the function $x^\mu(f(t))$ will describe the same boundary curve C . As was emphasized in [25], for a general curve we have

$$X_\mu^\psi(s, 0) = x_\mu(t(s)) \quad (25)$$

where the presence of a reparametrizing function $t(s)$ ($t'(s) \geq 0$) is required at the classical level to fulfill the conformal gauge (or the Virasoro constraints) and $t(s)$ depends upon the curve C . Equation (3) is of this type for the proper-length parametrization. $t(s)$ is related to the boundary value $\psi(s)$ of the Liouville field as

$$t'(s) = e^{\psi(s)/2 - \psi(t(s))/2} \quad (26)$$

(notice that $\psi(s) = 0$ for the proper-length parametrization). At the quantum level we have to path integrate over $t(s)$ [39], which is required for the consistency of the Polyakov string formulation. It is basically the same as the path-integration over $\psi(z)$ in Eq. (8).

Substituting Eq. (25) into the right-hand side of Eq. (24), we can rewrite the boundary action as Douglas' integral

$$S_B = \frac{1}{4\pi} \int_{-\infty}^{+\infty} ds_1 ds_2 \frac{[x(t(s_1)) - x(t(s_2))]^2}{(s_1 - s_2)^2}, \quad (27)$$

whose minimization with respect to $t(s)$ determines the minimal surface spanned by the curve C (or the classical string configuration $X_\mu^\psi(z)$), i.e. the choice of $\psi(s)$ which ensures that also (6) is satisfied.

The classical boundary action for the rectangle can be evaluated by substituting Eqs. (11) and (13) into Eq. (24). We then find

$$\begin{aligned} & \frac{1}{2} \int d^2 z \partial_a X_\mu^\psi \cdot \partial_a X_\mu^\psi \\ &= \frac{1}{2} \left[T^2 \frac{K(\sqrt{1-r})}{K(\sqrt{r})} + R^2 \frac{K(\sqrt{r})}{K(\sqrt{1-r})} \right] \end{aligned} \quad (28)$$

and, since at the classical level r is linked to T/R by Eq. (15), we recover the area RT . However, if we want to perform the functional integrations over reparametrizations of the boundary we cannot assume Eq. (15) and r will become an integration parameter as will be discussed below.

D. The one-loop calculation

We now have to perform the functional integration over ϕ for a fixed ψ and then over ψ in (8). In performing the integral over ψ we will expand to quadratic approximation around the parametrization which lead to (28) with r as a free parameter. Since the integration of ϕ represents the same order as that over ψ , it is consistent to keep the boundary condition of ϕ also to be the one which leads to (28), i.e. to be independent of the quadratic fluctuations of ψ . The two calculations have already been performed, the ϕ integration in [38] and the ψ integral in [41]. Here we basically combine the two results and for consistency add a few details. First we address the ϕ integral, next the ψ integral.

1. Lüscher term in UHP coordinates

In the world sheet parametrization one naturally expands around $\phi = 0$.³ Mapping the $\omega \in \omega_R \times \omega_T$ to the upper half-plane we then expand around

$$\phi_{cl} = 2 \log \left| \frac{\partial \omega(z)}{\partial z} \right|, \quad (29)$$

where the map $\omega(z)$ was explicitly given above. We thus split the Liouville field into the classical and quantum parts: $\phi = \phi_{cl} + \phi_q$. Then we notice that ϕ_q vanishes at the boundary since the boundary condition is already satisfied by ϕ_{cl} . The path integral over ϕ_q decouples in the expansion about the saddle point to the quadratic approximation and shifts the dimension by 1 just as for the closed string [43–45]. The ϕ part of the integration in (8) can thus, to quadratic order, be written as

$$\int \mathcal{D}\phi(z) e^{\frac{d-26}{96\pi} \int d^2 z \partial_a \phi \partial_a \phi} = e^{\frac{d-25}{96\pi} \int d^2 z \partial_a \phi_{cl} \partial_a \phi_{cl}}. \quad (30)$$

This shift seems unavoidable when using the Polyakov formulation.

The computation of the integral

$$\frac{1}{24\pi} \int d^2 z \partial_a \log \left| \frac{\partial \omega(z)}{\partial z} \right| \partial_a \log \left| \frac{\partial \omega(z)}{\partial z} \right|, \quad (31)$$

³ The derivation of the Lüscher term for the Polyakov string in the world sheet parametrization was given in Ref. [42].

simplifies, if we set $s_1 = 0$, $s_2 = r$ ($0 < r < 1$), $s_3 = 1$ and $s_4 = \infty$ in the expression for $\omega(z)$. Then

$$\omega(z) = \int_r^z \frac{ds}{\sqrt{(1-s)(s-r)s}} \quad (32)$$

and

$$\begin{aligned} \sqrt{g_{\text{cl}}(x, y)} &= \frac{RT}{4K(\sqrt{r})K(\sqrt{1-r})} \\ &\times \frac{1}{\sqrt{((x-1)^2 + y^2)((x-r)^2 + y^2)(x^2 + y^2)}}. \end{aligned} \quad (33)$$

Here we have used Eq. (13) for ω_R and ω_T .

Modulo an inessential (logarithmic) divergence at large y we find that (31) is given by

$$\begin{aligned} &-\frac{1}{96} \log [r^2(1-r)^2 \epsilon_1 \epsilon_2 \epsilon_3] + \text{const.} \\ &= -\frac{1}{24} \log [r(1-r)] + \text{const.}, \end{aligned} \quad (34)$$

where we have substituted Eq. (21) for ϵ_i 's. Using the asymptotes

$$K(\sqrt{r}) \xrightarrow{r \rightarrow 1} \frac{1}{2} \log \frac{16}{1-r}, \quad K(\sqrt{1-r}) \xrightarrow{r \rightarrow 1} \frac{\pi}{2}, \quad (35)$$

we reproduce the standard Lüscher term (per one degree of freedom)

$$\frac{1}{24} \log \frac{16}{1-r} = \frac{\pi \omega_T}{24 \omega_R} \quad (36)$$

for $\omega_T \gg \omega_R$.⁴

While the result (34) is derived for the particular regularization introduced in Subsect. IIB, we believe it is universal, as the Lüscher term is.

⁴ In deriving Eq. (34) we have not assumed that r is near 0 or 1, so it can be arbitrary. In world sheet coordinates we are expanding around $\phi = 0$ and thus simply calculating the determinant of a Laplacian. This determinant is a product over the eigen modes of the Laplacian and can be expressed [46] through the Dedekind η -function:

$$\prod_{m,n=1}^{\infty} \left(\frac{\pi m^2}{\omega_T^2} + \frac{\pi n^2}{\omega_R^2} \right) = \frac{1}{\sqrt{2\omega_R}} \eta \left(i \frac{\omega_T}{\omega_R} \right).$$

We have numerically verified with Mathematica that indeed

$$\frac{1}{2(K(\sqrt{1-r}))^{1/2}} \eta \left(i \frac{K(\sqrt{1-r})}{K(\sqrt{r})} \right) = \frac{1}{2^{5/6} \pi^{1/2}} [r(1-r)]^{1/12}$$

in the whole range $0 \leq r \leq 1$. It may be interesting to know whether this identity is known to the Math community.

2. The boundary reparametrization path integral

We finally have to perform the integral over quadratic fluctuations of the boundary field ψ in (8) (the main reason we introduced the UHP). As was shown in [41], accounting for reparametrizations in a quadratic approximation (justified by long strings) results for an outstretched rectangle in Eq. (36) with the extra factor of 24. The computation is sketched in Appendix A. Adding the contribution to the effective action from the ϕ and the ψ integrations in (8) leads to the Lüscher term

$$\left(1 + \frac{d-25}{24} \right) \log \frac{1}{1-r} = \frac{d_{\perp}}{24} \log \frac{1}{1-r} \quad (37)$$

with $d_{\perp} = d - 1$.⁵

A subtlety in performing the integral over the ψ field is that we have to fix the projective $PSL(2, \mathbb{R})$ symmetry which is inherited by the Douglas' integral (27). Technically, it is convenient to split the group of reparametrizations of the rectangle onto reparametrizations at the edges of the rectangle with fixed corners, that is to fix

$$t(s_i) = s_i \quad \text{for } i = 1, 2, 3, 4. \quad (38)$$

These reparametrizations simply relabel the points of the same edge. Thanks to the projective symmetry, the result depends on only one parameter: the ratio r . Additionally, we have to reparametrize at four corners, that is to integrate over four s_i 's, leaving their order at the real axis. This gives the (infinite) volume of $PSL(2, \mathbb{R})$ times the integral over r , which in the semiclassical (or one-loop) approximation is saturated by the classical value determined by Eq. (15).

Adding (28) and (37) we finally obtain for $(1-r) \ll 1$ the following effective action

$$S_{\text{eff}} = \frac{1}{4\pi\alpha'} \left[T^2 \frac{\pi}{\log \frac{1}{1-r}} + R^2 \frac{\log \frac{1}{1-r}}{\pi} \right] - \frac{d_{\perp}}{24} \log \frac{1}{1-r}. \quad (39)$$

We emphasize again that the formula does not rely on r obeying Eq. (15) (but it assumes $(1-r) \ll 1$). For r not satisfying Eq. (15) the induced metric tensor is diagonal but not conformal (or isothermal).

Beyond the semiclassical approximation we have to integrate the exponential of (minus) (39) over r from 0 to 1. There are two important cases, where this integral has a saddle point: $T \gg R$ and large d . Then we can simply replace the integration by the minimization with respect to r . Technically, this reminds the derivation of

⁵ For the Polyakov string formulation the number d_{\perp} of fluctuating transverse (physical) degrees of freedom of the string, which the Lüscher term is proportional to, is $d-1$ rather than $d-2$ because the dimension is effectively shifted by 1, as is already mentioned, since the Liouville field also fluctuates [43–45]. We have $d_{\perp} = 2$ for the QCD string.

the Regge limit of the Veneziano amplitude. Such an analogy is considered in detail in Ref. [13].

The limit of large $|d|$ is generically the one, where a mean-field approximation becomes exact. We can therefore look at r as if it was a parameter of the variational mean-field ansatz. Its best description of an exact result is reached at the minimum. For $|d| \rightarrow \infty$ the minimum gives the exact energy of the string ground state, while for finite d we may generically expect an approximate value larger than the exact one. However, it can be argued for the Polyakov formulation of the Nambu–Goto string that we may expect to obtain the exact result for $T \gg R$ even at finite d by the mean field.

E. Spectrum of pure Nambu–Goto

To compute the spectrum, we have to minimize, as is shown in the previous Subsection, the effective action (39) with respect to r . The saddle-point equation is quadratic in $\log(1-r)$ and has the solution

$$\frac{\log \frac{1}{1-r_*}}{\pi} = \frac{T}{\sqrt{R^2 - R_0^2}}, \quad (40)$$

where R_0^2 is the inverse tachyon mass squared

$$\begin{aligned} R_0^2 &= \frac{\pi^2 d_\perp}{6} \alpha' && \text{for open string,} \\ R_0^2 &= \frac{2\pi^2 d_\perp}{3} \alpha' && \text{for closed string} \end{aligned} \quad (41)$$

with the periodic boundary condition⁶ along the 1-direction. This replaces the pure classical Eq. (15). At the minimum we have

$$S_{\text{eff}} = \frac{T}{2\pi\alpha'} \sqrt{R^2 - R_0^2}, \quad (42)$$

reproducing the Alvarez–Arvis energy [26, 27] of the ground state. Our derivation extends to UHP that of [13] given for the world sheet coordinates.

For the average of the induced metric tensor we have Eq. (10) with

$$\begin{aligned} \rho &= \frac{\bar{\rho}}{\sqrt{((x-1)^2 + y^2)((x-r)^2 + y^2)(x^2 + y^2)}}, \\ \bar{\rho} &= \frac{R^2 - R_0^2/2}{\omega_R^2}, \end{aligned} \quad (43)$$

which is fulfilled owing to Eq. (40). It replaces the classical Eq. (33), which is recovered as $\alpha' \rightarrow 0$. For the ratio

of the area of a typical surface to the minimal area we then find

$$\frac{\langle A \rangle_\phi}{RT} = \frac{R^2 - R_0^2/2}{R\sqrt{R^2 - R_0^2}}. \quad (44)$$

It tends to 1 as $R \rightarrow \infty$, but blows up near the tachyonic singularity at $R = R_0$, when crumpling of surfaces occurs. Then we can no longer trust in the mean-field approximation.

III. STRING WITH EXTRINSIC CURVATURE

A simplest generalization of the Nambu–Goto string is bosonic string with an extrinsic curvature, which is the next-order operator after the Nambu–Goto action. It was first introduced for QCD string in [31, 32]. The original idea was that it provides rigidity of the string that makes it smoother. The spectrum of such a rigid string is investigated in [33–35] (for a review see Ref. [36]). We briefly review some of these results below.

The action of the bosonic (Polyakov) string with the extrinsic curvature term reads

$$S_{\text{r.s.}} = \frac{K}{2} \int d^2z \sqrt{g} g^{ab} \partial_a X \cdot \partial_b X + \frac{1}{2\alpha} \int d^2z \sqrt{g} \Delta X \cdot \Delta X, \quad (45)$$

where $K = 1/2\pi\alpha'$ is the string tension, α is a dimensionless constant (rigidity) and Δ is the 2d Laplace–Beltrami operator.

It is custom to consider g_{ab} as an induced metric and, introducing $\rho = \sqrt{g}$ and the Lagrange multipliers λ^{ab} , to rewrite the action (45) in the conformal gauge as

$$\begin{aligned} S_{\text{r.s.}} &= K \int d^2z \rho + \frac{1}{2\alpha} \int d^2z \frac{1}{\rho} \partial^2 X \cdot \partial^2 X \\ &\quad + \frac{1}{2} \int d^2z \lambda^{ab} (\partial_a X \cdot \partial_b X - \rho \delta_{ab}). \end{aligned} \quad (46)$$

A. World sheet parametrization

Likewise for the Nambu–Goto string, we consider the world sheet parametrization and restrict ourselves with a mean-field (variational) ansatz, when only X^\perp fluctuates. It is exact at large d for $K \sim d$ and $\alpha \sim 1/d$. We write (for $\omega_T = T$)

$$\begin{aligned} X_{\text{mf}}^1(\omega) &= \frac{\omega_1}{\omega_R} R, & X_{\text{mf}}^2(\omega) &= \omega_2, & X^\perp(\omega) &= \delta X^\perp(\omega), \\ \rho_{\text{mf}}(\omega) &= \rho, & \lambda_{\text{mf}}^{11}(\omega) &= \lambda^{11}, & \lambda_{\text{mf}}^{22}(\omega) &= \lambda^{22}, \\ \lambda_{\text{mf}}^{12}(\omega) &= \lambda_{\text{mf}}^{21}(\omega) &= 0 \end{aligned} \quad (47)$$

⁶ The Lüscher term is then four times larger than for an open string.

and

$$\begin{aligned} \frac{1}{T} S_{\text{mf}} &= \frac{1}{2} \left(\lambda^{11} \omega_R + \lambda^{22} \frac{R^2}{\omega_R} \right) + \rho \left(K - \frac{\lambda^{11}}{2} - \frac{\lambda^{22}}{2} \right) \omega_R \\ &\quad + \frac{d}{2T} \text{tr} \log \left(-\frac{\lambda^{11}}{\rho} \partial_1^2 - \frac{\lambda^{22}}{\rho} \partial_2^2 + \frac{1}{\alpha \rho^2} (\partial_1^2 + \partial_2^2)^2 \right). \end{aligned} \quad (48)$$

The determinant in the last line of Eq. (48) can be

evaluated using the formulas of Refs. [33–35]. Using a momentum-space representation, integrating over dk_2 (as $T \rightarrow \infty$), regularizing via the zeta function and introducing

$$\Xi = \frac{\sqrt{\alpha \rho \lambda^{11}} \omega_R}{2\pi} \quad (49)$$

instead of ρ , we get

$$\begin{aligned} \frac{1}{T} S_{\text{mf}} &= \frac{1}{2} \left(\lambda^{11} \omega_R + \lambda^{22} \frac{R^2}{\omega_R} \right) + \left(\frac{2K(\mu)}{\lambda^{11}} - 1 - \frac{\lambda^{22}}{\lambda^{11}} \right) \frac{2\pi^2 \Xi^2}{\alpha(\mu) \omega_R} \\ &\quad + \frac{2\pi d}{\omega_R} \left\{ -\frac{1}{6} + \frac{\Xi}{2} + \sum_{n \geq 1} \left[\sqrt{\frac{\Xi^2}{2} + n^2 + \Xi \sqrt{\frac{\Xi^2}{4} + \left(1 - \frac{\lambda^{22}}{\lambda^{11}}\right) n^2}} \right. \right. \\ &\quad \left. \left. + \sqrt{\frac{\Xi^2}{2} + n^2 - \Xi \sqrt{\frac{\Xi^2}{4} + \left(1 - \frac{\lambda^{22}}{\lambda^{11}}\right) n^2} - 2n - \frac{\Xi^2}{4n} \left(1 + \frac{\lambda^{22}}{\lambda^{11}}\right)} \right] \right\}. \end{aligned} \quad (50)$$

In Eq. (50) we have introduced an ultraviolet cutoff a_{UV} by

$$\sum_{n \geq 1} \frac{1}{n} = \log \frac{1}{\mu a_{\text{UV}}}. \quad (51)$$

and performed the renormalization of the parameters K and α of the bare action by introducing (renormalized)

$$\alpha(\mu) = \frac{\alpha}{1 - \frac{\alpha d}{4\pi} \log \frac{1}{\mu a_{\text{UV}}}}, \quad K(\mu) = K \frac{\alpha(\mu)}{\alpha} \quad (52)$$

as is prescribed by asymptotic freedom of the model [31, 32]. Then UV divergences disappear in Eq. (50) and the result is finite.

As $R \rightarrow \infty$ the action (50) can be minimized iteratively, order by order in α'/R^2 , which is nothing but a semiclassical expansion, starting from the classical approximation:

$$\omega_R = R, \quad \lambda^{11} = \lambda^{22} = K, \quad \rho = 1. \quad (53)$$

Because $\lambda^{11} = \lambda^{22}$ order by order of the expansion in α'/R^2 or, correspondingly, in $1/\Xi^2$, the sum over n in Eq. (50) is exponentially suppressed [35]. We thus reproduce the same solution as for the Nambu–Goto string to any order in α'/R^2 .

However, the extrinsic curvature becomes important at $\alpha'/R^2 \sim 1$. The case of small α can be analyzed analytically. Then the extrinsic curvature dominates over

the Nambu–Goto term and great simplifications occur in Eq. (50). For small α we have Ξ also small and the determinant in the last line in Eq. (48) equals

$$\frac{d}{2T} \text{tr} \log(\dots) \longrightarrow -\frac{\pi d}{3\omega_R} + \frac{d}{2} \sqrt{\alpha \rho \lambda^{11}} \quad (\text{closed string}) \quad (54)$$

as $\alpha \rightarrow 0$. The first term on the right-hand side is twice larger than usual Lüscher's term, because the quartic operator dominates over quadratic that doubles degrees of freedom. The second term comes from zero modes.

For small α we can easily minimize the mean-field action (48) to obtain [37]

$$\begin{aligned} E_0 &= \lambda^{11} \omega_R, \\ \sqrt{\lambda^{11}} &= \frac{3}{8} \frac{d\sqrt{\alpha}}{R} + \sqrt{\frac{9}{64} \frac{d^2 \alpha}{R^2} + K - \frac{\pi d}{3R^2}}, \\ \omega_R &= \sqrt{R^2 - \frac{dR}{2} \sqrt{\frac{\alpha}{\lambda^{11}}}} \end{aligned} \quad (55)$$

and

$$\rho = \frac{R^2}{\omega_R^2}. \quad (56)$$

These formulas are applicable for small α if

$$R^2 \ll \frac{\alpha'}{\alpha} \quad (57)$$

when Ξ is small.

B. UHP parametrization

With the extrinsic-curvature term included, the contribution of quantum fluctuations to the effective action is given by the determinant in Eq. (48) which is not computable by the Seeley expansion like in the massive case (see Appendix B for a review). It can be easily evaluated, however, for either large or small α . For large α we can neglect the quartic operator, reproducing the above results for the conformal anomaly. For small α we can instead neglect the quadratic operator, so the result is simply twice larger than computed above. This factor of 2 is simply related with doubling of degrees of freedom in the conformal anomaly, which produces twice larger Lüscher's term.

We thus obtain at large d

$$\frac{d}{2T} \text{tr} \log \left(-\frac{\lambda^{11}}{\rho} \partial_1^2 - \frac{\lambda^{22}}{\rho} \partial_2^2 + \frac{1}{\alpha \rho^2} (\partial_1^2 + \partial_2^2)^2 \right) = \begin{cases} -\frac{\pi d}{6\omega_R} \sqrt{\frac{\lambda^{22}}{\lambda^{11}}} & \alpha \rightarrow \infty \\ -\frac{\pi d}{3\omega_R} & \alpha \rightarrow 0 \end{cases}. \quad (58)$$

The saddle-point minimization in the case $\alpha \rightarrow \infty$ is like as in Ref. [26] and gives [cf. Eq. (43)]

$$\lambda^{11} = \lambda^{22} = K, \quad \omega_R = \sqrt{R^2 - \frac{\pi d}{3K}}, \quad \bar{\rho} = \frac{1 - \frac{\pi d}{6KR^2}}{1 - \frac{\pi d}{3KR^2}}, \quad (59)$$

reproducing Eq. (42).

The saddle-point minimization in the case $\alpha \rightarrow 0$ is like as in Ref. [37] and gives

$$\lambda^{11} = K - \frac{\pi d}{3R^2}, \quad \lambda^{22} = K + \frac{\pi d}{3R^2}, \quad \omega_R = R, \quad \bar{\rho} = 1. \quad (60)$$

For the energy of the ground state we obtain

$$E_0(R) = KR - \frac{\pi d}{3R} \quad (61)$$

to be compared with Eq. (42) for the iterative solution. We see that the semiclassical one-loop result is exact as $\alpha \rightarrow 0$. This saddle point is simply the $\alpha \rightarrow 0$ limit of (55).

The domain (57), where Eqs. (60), (61) are applicable, overlaps with the domain $\alpha' \ll R^2$, where the iterative solution coincides with the Nambu-Goto one modulo exponential terms. However, the ground state energy (61) is smaller than (42), so it is the solution (60) that is realized for $R^2 \ll \alpha'/\alpha$. We emphasize once again that Eq. (61) applies *only* in the domain (57). For large values of $R^2 \gtrsim 1/K\alpha$ the iterative solution (42) applies.

From Eqs. (60) we deduce that

$$\frac{\langle A \rangle_\phi}{RT} = 1 \quad (62)$$

as $\alpha \rightarrow 0$ in contrast to Eq. (44). We see that the string indeed becomes rigid and behaves as a stick as $\alpha \rightarrow 0$. Because the ground-state energy (61) is lower than (42), the string behaves as rigid for $R^2 \ll 1/K\alpha$ and as the usual Nambu-Goto string for $R^2 \gg 1/K\alpha$. The change between the regimes (42) and (61) cannot be seen, therefore, within the $1/R$ -expansion.

The fact that the dependence (61) is not seen in the rather precise lattice Monte-Carlo spectrum [8] (see [47] for a review and references therein) probably means that a large rigidity term is ruled out for QCD string.

IV. EXTERNAL GEOMETRY OF EFFECTIVE STRING

Trajectories $X^\mu(z)$ of strings moving in d -dimensional space are surfaces embedded in Euclidean R^d space. Therefore, the differential geometry of the string world sheet is totally defined by the induced metric $g_{ab} = \partial_a X_\mu \partial_b X^\mu$ and second quadratic forms $h_{ab}^j = n_{\mu}^j \nabla_a \partial_b X^\mu$ and $H_a^{ij} = n_{\mu}^i \partial_a n^{j,\mu}$, where $j = 1, \dots, d-2$ counts coordinates transversal to the surface. For immersed surfaces, which means that the induced metric g_{ab} is not singular, the first and second fundamental forms should fulfill the Peterson-Codazzi equations

$$\begin{aligned} \mathcal{R}_{dacb} &= h_{ab}^j h_{cd}^j - h_{ac}^j h_{bd}^j, \\ \epsilon^{ab} \partial_a H_b^{ij} &= \epsilon^{ab} H_a^{ik} H_b^{kj} + \epsilon^{ab} h_a^{i,c} h_{bc}^j, \\ \epsilon^{ab} \nabla_a h_{bc}^i &= \epsilon^{ab} h_{ac}^j H_b^{ji}, \end{aligned} \quad (63)$$

where the summation over repeating indexes i, j, a, b, \dots is implied. These equations take place because the tangent vectors $\partial_a X^\mu$, $a = 1, 2$ are derivatives and together with the normal vectors n_{μ}^j they form a complete set of basic vectors in the target space of embedding, namely $\partial_a X^\nu \partial^a X^{\nu'} + n^{j,\nu} n_{j,\nu'} = \delta^{\nu\nu'}$. In two dimension, due to the symmetry properties of the 4-range curvature tensor \mathcal{R} over its indexes, all the information reduces to the scalar Gaussian curvature R

$$\mathcal{R}_{dacb} = \frac{R}{2} (g_{ab} g_{dc} - g_{ad} g_{bc}). \quad (64)$$

Therefore, we can reduce Gaussian curvature R to the second fundamental h-forms. Moreover, due to the definitions of the fundamental forms g_{ab} , h_{ab}^j , H_a^{ij} and the Peterson-Codazzi equations, all scalars of the surface coordinates X^μ , as well as their higher covariant derivatives, can be always reduced to scalar polynomials of the second fundamental forms h and H .

We suggest that the dynamics of strings is defined by a quadratic elliptic operator \mathcal{D} , which may explicitly depend on external geometry of surfaces and is reparametrization invariant.

As it follows from the work [25], any 2d gravity theory will always have a quantum anomaly of conformal transformations $g_{ab} \rightarrow \rho g_{ab}$, unless some cancellation between bosons, fermions and ghosts will take place at certain

critical dimensions. The anomaly expresses itself in the fact that, though the trace of the stress-energy tensor is zero at the classical level, its quantum average is nonzero and is connected with the expansion of the heat-kernel operator $e^{-\mathcal{D}/\mu^2}$, where μ^2 is a regularization scale of quantum fluctuations. Following the works [25, 40], we expect that each central charge 1 will give the following contribution to the trace of the stress-energy tensor

$$\begin{aligned} \langle T_a^a \rangle = \langle e^{-\mathcal{D}/\mu^2} \rangle &= \frac{1}{24\pi} \sqrt{g} R(\xi_a) + \frac{c_0}{16\pi} \sqrt{g} (h^j)^2 \\ &+ \mu^2 \sqrt{g} + \mu^2 \sqrt{g} F \left[\frac{1}{\mu^2} h_{ab}^i K \left[\frac{1}{\mu^2} H^2 \right]^{ij} h_{cd}^j \right]. \end{aligned} \quad (65)$$

Here $g = \det[g_{ab}]$, $R(\xi_a)$ and $h^j = g^{ab} h_{ab}^j$ ($j = 1, \dots, d-2$) are the Gaussian and mean sectional curvatures, respectively, and

$$K \left[\frac{1}{\mu^2} H^2 \right] = 1 + \frac{k_2}{\mu^2} H_{s_1} H_{s_2} + \dots + \frac{k_{2m}}{\mu^{2m}} H_{s_1} \dots H_{s_{2m}} \dots \quad (66)$$

is an even-order series of the matrices H_s^{ij} , while $F[\frac{1}{\mu^2} h_{ab}^j h_{cd}^j]$ is a scalar reparametrization-invariant series of its tensor arguments $\frac{1}{\mu^2} h_{ab}^j h_{cd}^j$, which starts from a square. Symbolically, we can write

$$F \left[\frac{1}{\mu^2} h_{ab}^j h_{cd}^j \right] = P_2 \left[\frac{1}{\mu^2} h_{ab}^j h_{cd}^j \right]^2 + P_3 \left[\frac{1}{\mu^2} h_{ab}^j h_{cd}^j \right]^3 + \dots \quad (67)$$

The notations P_k , $k = 2, 3, \dots$, in this expression mean scalar polynomials of k 'th order of homogeneity, which are obtained by all type of contractions of the indexes in $h_{a_1 b_1}^{j_1} h_{c_1 d_1}^{j_1} \otimes \dots \otimes h_{a_k b_k}^{j_k} h_{c_k d_k}^{j_k}$ by using the metric g^{ab} . As we see below, the terms $\sum_{j=1}^{d-2} h_{ab}^j h_{cd}^j$ produce the coefficients of $d-2$, while an insertion of H^{ij} does not change it, but increases the order of $1/\mu^2$. Therefore, we should drop all H^{ij} 's to the leading order in $(d-2)/\mu^2$ and take $K[\frac{1}{\mu^2} H^2] = 1$.

The expansion of $\langle e^{-\mathcal{D}/\mu^2} \rangle$, that enters Eq. (65), in the powers of μ^{-2k} is known as the Seeley expansion. It is a standard way of calculations for quadratic elliptic operators \mathcal{D} as presented in Refs. [39, 40, 48]. The first term in the expression (65) is usual and defined by internal geometry of the string world sheet (i.e. the metric), while the second term comes from external geometric characteristics of the surfaces. For example, for the Green-Schwartz superstring with $N=1$ supersymmetry it was obtained that $c_0 = 1$ [28]. These terms are universal and their coefficients are dimensionless. Other terms have dimensionfull coefficients and therefore break conformal invariance explicitly.

Equation (65) for the conformal anomaly allows us to find out the most general form of the effective action \mathcal{L}_{eff} . Namely, we should integrate the Word identity

$$\delta W = -\frac{\delta \rho}{2\rho} \langle T_a^a \rangle, \quad (68)$$

where $\langle T_a^a \rangle$ in (65) should be written in a conformal gauge with the metric $g_{ab} = \rho \delta_{ab}$. In order to do that, let us first express $h_{ab}^j h_{cd}^j$ as a sum of the tensors

$$h_{ab}^j h_{cd}^j = \frac{1}{2} h_{a\{b}^j h_{c\}d}^j - \frac{1}{2} \mathcal{R}_{adbc}, \quad (69)$$

symmetrized and antisymmetrized over the indexes b, c . Here the curly brackets $\{ \}$ around the b, c indexes mean a symmetrization over them. Substituting this representation into the expansion (67) and using (64), we obtain

$$\begin{aligned} F \left[\frac{1}{\mu^2} h_{ab}^j h_{cd}^j \right] &= \tilde{P}_2 \left[\frac{1}{\mu^2} h_{a\{b}^j h_{c\}d}^j \right]^2 Q_2[\sqrt{g}R] \\ &+ \tilde{P}_3 \left[\frac{1}{\mu^2} h_{a\{b}^j h_{c\}d}^j \right]^3 Q_3[\sqrt{g}R] + \dots, \end{aligned} \quad (70)$$

where $Q_p[R]$ are the series in the Gaussian curvature R

$$Q_p[\sqrt{g}R] = 1 + w_{p1} \sqrt{g}R + \dots + w_{ps} \sqrt{g}^s R^s + \dots \quad (71)$$

In the conformal gauge $R = -\frac{2}{\rho} \partial_a^2 \log[\rho]$, while

$$h_{a\{b}^j h_{c\}d}^j = \rho (\partial_a \vec{n}^j \vec{e}_{\{b}^j) (\partial_d \vec{n}^j \vec{e}_{c\}}^j). \quad (72)$$

A conformal variation of this expressions gives $\delta(\sqrt{g}R) = -2\partial_a^{-2} \frac{\delta \rho}{\rho}$ and $\delta h_{a\{b}^j h_{c\}d}^j = \frac{\delta \rho}{\rho} h_{a\{b}^j h_{c\}d}^j$, which yields

$$\begin{aligned} \frac{\delta \rho}{\rho} [\sqrt{g}R]^n &= -\frac{1}{2(n+1)} \delta [\sqrt{g}R \partial^{-2} (\sqrt{g}R)^n], \\ \frac{\delta \rho}{\rho} [h_{a\{b}^j h_{c\}d}^j]^n &= \frac{1}{n} \delta [h_{a\{b}^j h_{c\}d}^j]^n. \end{aligned} \quad (73)$$

The relations presented above allow us to integrate the Word identity (68) and to calculate the effective action

$$\begin{aligned} \mathcal{L}_{eff} &= -\frac{c}{96\pi} \sqrt{g} [R \Delta^{-1} R + \mu^2] - \frac{c_0 c}{16\pi} \sqrt{g} [h_a^{j,a}]^2 \Delta^{-1} R \\ &- c \sqrt{g} F_1 \left[\frac{1}{\mu^2} h_{ab}^j h_{cd}^j, \sqrt{g}R \right] \\ &- c \sqrt{g} \sqrt{g} R \partial^{-2} F_2 \left[\frac{1}{\mu^2} h_{ab}^j h_{cd}^j, \sqrt{g}R \right], \end{aligned} \quad (74)$$

where c is the central charge of the system, while $F_1[\cdot]$ and $F_2[\cdot]$ are defined by $F[\cdot]$ as

$$\begin{aligned} F_1 \left[\frac{1}{\mu^2} h_{ab}^j h_{cd}^j \right] &= \frac{1}{2} \tilde{P}_2 \left[\frac{1}{\mu^2} h_{ab}^j h_{cd}^j \right]^2 Q_2[\sqrt{g}R] \\ &+ \frac{1}{3} \tilde{P}_3 \left[\frac{1}{\mu^2} h_{ab}^j h_{cd}^j \right]^3 Q_3[\sqrt{g}R] + \dots, \\ F_2 \left[\frac{1}{\mu^2} h_{ab}^j h_{cd}^j \right] &= \tilde{P}_2 \left[\frac{1}{\mu^2} h_{ab}^j h_{cd}^j \right]^2 \tilde{Q}_2[\sqrt{g}R] \\ &+ \tilde{P}_3 \left[\frac{1}{\mu^2} h_{ab}^j h_{cd}^j \right]^3 \tilde{Q}_3[\sqrt{g}R] + \dots \end{aligned} \quad (75)$$

Here

$$\begin{aligned} \tilde{Q}_p[\sqrt{g}R] &= 1 - \frac{1}{2 \times 2} w_{p1} \sqrt{g}R \\ &+ \dots - \frac{1}{2(s+1)} w_{ps} \sqrt{g}^s R^s + \dots \end{aligned} \quad (76)$$

The functions $F_1[\cdot]$ and $F_2[\cdot]$ are model dependent. The coefficients of their expansion in $h_a^{j,a} h_b^{j,b}$ and R depend essentially on what theory we are considering: various types of superstrings or gauge theories.

If we have in the effective action additional operators of the type in Eq. (74), their correction to the spectrum can be computed by evaluating their values for the classical solution (11), (13) with r substituted by r_* given by Eq. (40). This is rigorous at least for large d , because the operator is expected to give a $1/d$ correction to the spectrum. Since $h_{ab}^j = 0$ and $R = 0$ for the classical solution (11), these additional terms in the effective action (74) do not change the spectrum. As we have seen in the previous Section, the spectrum may change, however, if the string length is not too large, because the first term in the effective action (74) may change.

We have ignored above in this Section possible boundary terms in the effective action, which may be required by consistency, like the well-known Gibbons–Hawking term in the gravitational action for an open manifold. The boundary terms play indeed the important role for the usual conformal anomaly (given by the first term on the right-hand side of Eq. (74)), as is already discussed, but apparently not for the other terms. We expect those vanish because the second fundamental form h_{ab}^j vanishes for the classical solution (11). We thus believe that the possible boundary terms will not effect the conclusion of the previous Paragraph, while this issue deserves a more thorough consideration.

V. CONCLUSIONS

We have considered QCD string as an effective string formed by fluxes of the gluon field. Its effective action, obtained by path integrating over short-range fluctuations, describes long-range stringy fluctuations. We have used the Polyakov formulation of an open string, where the target-space coordinates X_μ and the world sheet metric g_{ab} are treated as independent variables. We have shown that the effective action has to be minimized with respect to g_{ab} in two cases: the limit of long strings and/or the large number d of space dimensions. This determines the ground-state energy of the string as a function of its length. We have found that the spectrum of the pure Nambu–Goto string is given by the Alvarez–Arvis formula (42). We have then added the next-relevant after Nambu–Goto operator in the infrared – the extrinsic curvature – and have shown that the spectrum is not changed order by order in the inverse string length, but is changed at intermediate distances (see Eq. (61) valid for very large rigidity). We have considered a most general effective action in the form of the conformal anomaly, that includes external geometry, and argued that the spectrum behaves similarly.

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Appendix A: Reparametrization path integral: rectangle

We briefly reproduce in this Appendix the computation [41] of the reparametrization path integral for a rectangle, using a different mapping of UHP onto rectangle.

Expanding about the minimizing function $s_*(t)$:

$$s(t) = s_*(t) + \sqrt{\frac{2\pi\alpha'}{RT}} \beta(t), \quad (\text{A1})$$

we write for the action to the quadratic order in β [41, 49]:

$$S_2[\beta] = \frac{1}{4\pi\omega_R\omega_T} \int ds_1 ds_2 \frac{|\omega'(s_1)| |\omega'(s_2)|}{(s_1 - s_2)^2} \times [\beta(t_*(s_1)) - \beta(t_*(s_2))]^2. \quad (\text{A2})$$

The minimizing function $t_*(s)$ is determined by the equation

$$\begin{aligned} \int_r^s \frac{dx}{\sqrt{(1-x)(x-r)x}} &= \omega_T \frac{t}{(1-t^2)} + \frac{\omega_R}{2} \\ &\text{for } r < s < 1, \\ \int_1^s \frac{dx}{\sqrt{(x-1)(x-r)x}} &= \omega_R \frac{(t^2-1)}{4t} + \frac{\omega_T}{2} \\ &\text{for } s > 1, \end{aligned} \quad (\text{A3})$$

so that

$$t_*(1) = \frac{\sqrt{\omega_T^2 + \omega_R^2} - \omega_T}{\omega_R} \xrightarrow{\omega_T \gg \omega_R} \frac{\omega_R}{2\omega_T}. \quad (\text{A4})$$

It is clear that the domain $0 < (s_1 - 1), (s_2 - 1) \ll 1$ is essential in the integral in Eq. (A2) for $(1-r) \ll 1$. It produces a large contribution $\sim (1-r)^{-2}$. To see this, we expand

$$\beta(t_*(s_2)) - \beta(t_*(s_1)) = \beta'(t_*(s_1)) t'_*(s_1) (s_2 - s_1). \quad (\text{A5})$$

We thus obtain (disregarding $\log(1-r)$'s)

$$\begin{aligned} S_2[\beta] &= \frac{1}{4\pi\omega_R\omega_T} \int ds_1 ds_2 \frac{[\beta'(t_*(s_1)) t'_*(s_1)]^2}{\sqrt{|s_1-1||s_1-r||s_2-1||s_2-r|}} \\ &\propto \int ds_1 ds_2 \frac{[\beta'(t_*(s_1))]^2}{\sqrt{|s_1-1||s_1-r|}} \frac{1}{(|s_2-1||s_2-r|)^{3/2}} \\ &\propto \frac{1}{(1-r)^2} \int dt [\beta'(t)]^2 \end{aligned} \quad (\text{A6})$$

because in this domain

$$t'_*(s) = \frac{4t_*(1)^2}{\omega_R \sqrt{(s-1)(s-r)}}, \quad (\text{A7})$$

that determines the boundary metric. It has to be regularized by slightly moving the boundary into UHP, as is already discussed in Subsect. II B. In passing from the first to the second line of Eq. (A6), we have also set $t'_*(s_1) = t'_*(s_2)$ with the given accuracy.

Computing the path integral over $\beta(t)$ by the standard mode expansion, obeying

$$\beta(0) = \beta(r) = \beta(1) = 0 \quad (\text{A8})$$

to get rid of the projective symmetry, we obtain with the aid of the zeta-function regularization

$$\begin{aligned} \int \mathcal{D}\beta(t) e^{-S_2[\beta]} &= \prod_{\text{modes}} [r(1-r)]^2 = \left[\frac{1}{\sqrt{r(1-r)}} \right]^2 \\ &= e^{-\log[r(1-r)]}, \end{aligned} \quad (\text{A9})$$

where we have inserted r to reflect the symmetry $r \rightarrow (1-r)$ of the rectangle. The power of 2 is because of the two sets of the modes which contribute to the product: one for the interval $[0, r]$ and another for $[1, \infty]$, obeying the boundary condition (A8).

This result for the reparametrization path integral seems to be exact at large TR/α' , because we can then restrict ourselves by the quadratic in β approximation.

It might seem that the result of this Appendix for the reparametrization (= the boundary Liouville field) path integral is less solid than the results for the path integral over the bulk Liouville field of Subsect. II D 1, but this is not the case! We have simply shown here how the standard results for the critical bosonic string can be reproduced by the given method.

Appendix B: Review of the Seeley-expansion method

The action, describing dynamics of the Liouville field ϕ , emerges from the path integration over $X_\mu(x, y)$ (and the ghosts) due to ultraviolet divergences regularized by a cutoff. For smooth ϕ it is the conformal anomaly displayed in Eq. (8).

Generically, the contribution from $X_\mu(x, y)$ to the Liouville action comes from the (regularized) determinant of the 2d Laplacian:

$$e^{-S_L} = [\det(-e^{-\phi}\partial^2)]_{\text{Reg.}}^{-d/2}. \quad (\text{B1})$$

For the Pauli–Villars regularization we shall consider the ratio of determinants of the form

$$e^{-S_L} = \left[\frac{\det(-e^{-\phi}\partial^2)}{\det(-e^{-\phi}\partial^2 + M^2)} \right]^{-d/2}, \quad (\text{B2})$$

where M is a regulator mass.

The standard technique for computing such determinants (reviewed in this Appendix) is applicable for

$$\Lambda^2 e^\phi \gg 1, \quad \text{or} \quad M^2 e^\phi \gg 1, \quad (\text{B3})$$

when it results in the conformal anomaly.

The standard results for the (regularized) determinants of the 2d Laplacian are obtained by Seeley's expansion [39, 40]:

$$\begin{aligned} \text{tr} \log(-\Delta) \Big|_{\text{div}} &= -\frac{1}{4\pi} \left\{ \Lambda^2 \int_D -\sqrt{\pi} \Lambda \int_{\partial D} \right. \\ &\quad \left. + \frac{1}{3} \log \Lambda^2 \left[\int_D \frac{R}{2} + \int_{\partial D} k \right] \right\} \end{aligned} \quad (\text{B4})$$

for the divergent part and

$$\text{tr} \log(-\Delta) \Big|_{\text{fin}} = -\frac{1}{24\pi} \left[\int_D \frac{1}{2} R \phi + \int_{\partial D} k \phi \right] - \frac{1}{4\pi} \int_{\partial D} k \quad (\text{B5})$$

for the finite part. Here

$$k = -\frac{1}{2} n^a \partial_a \phi \quad (\text{B6})$$

is the geodesic curvature and n^a is the inward normal unit vector.

Equation (B5) for the finite part can be derived as follows. Let us apply the variational derivative $\delta/\delta\phi(z)$ to the (regularized) determinant

$$\text{tr} \log(-\Delta + M^2) \Big|_{\text{Reg}} = -\int_{\Lambda^{-2}}^{\infty} \frac{d\tau}{\tau} \text{tr} e^{\tau(\Delta - M^2)} \quad (\text{B7})$$

and represent the result as $\partial/\partial\tau$ plus an additional term. We then obtain

$$\begin{aligned} \frac{\delta}{\delta\phi(z)} \text{tr} \log(-\Delta + M^2) \Big|_{\text{Reg}} &= \langle z | e^{(\Delta - M^2)/\Lambda^2} | z \rangle \\ &\quad - M^2 \int_{\Lambda^{-2}}^{\infty} d\tau \langle z | e^{\tau(\Delta - M^2)} | z \rangle. \end{aligned} \quad (\text{B8})$$

For $M^2 = 0$ we can substitute the Seeley expansion of the heat kernel in the first term on the right-hand side [39, 40, 48]:

$$\langle z | e^{\Delta/\Lambda^2} | z \rangle = \Lambda^2 a_0 + \Lambda a_1 + a_2 \quad (\text{B9})$$

with

$$\begin{aligned} a_0 &= \frac{1}{4\pi}, \quad a_1 = -\frac{1}{8\sqrt{\pi}} \delta^{(1)}(z - z_B), \\ a_2 &= \frac{1}{12\pi} \left(\frac{1}{2} R + k \delta^{(1)}(z - z_B) \right) \end{aligned} \quad (\text{B10})$$

and reproduce the conformal anomaly on the right-hand side of Eq. (B5). For the second term we can use this

expansion only for large M^2 , when small τ are essential in the integral, to obtain

$$\frac{\delta}{\delta\phi(z)} \text{tr} \log(-\Delta + M^2) \Big|_{\text{Reg}} \stackrel{\text{large } M}{=} \Lambda^2 a_0 + \Lambda a_1 - M^2 a_0 - M^2 a_0 \log \frac{\Lambda^2}{M^2} - \frac{1}{2} M a_1. \quad (\text{B11})$$

The ratio in Eq. (B2) is analogously regularized as

$$\text{tr} \log \frac{-\Delta}{(-\Delta + M^2)} = - \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \text{tr} e^{\tau\Delta} (1 - e^{-\tau M^2}), \quad (\text{B12})$$

which is still logarithmically divergent at small τ because the Seeley expansion (B9) starts from the term proportional to $1/\tau$. For large M this is explicitly seen in Eq. (B11).

To get rid of this logarithmic divergence, we may consider the ratio

$$\mathcal{R}^{(2)} \equiv \frac{\det(-\Delta) \det(-\Delta + 2M^2)}{\det(-\Delta + M^2)^2}, \quad (\text{B13})$$

when

$$\text{tr} \log \mathcal{R}^{(2)} = - \int_0^{\infty} \frac{d\tau}{\tau} \text{tr} e^{\tau\Delta} (1 - e^{-\tau M^2})^2 \quad (\text{B14})$$

is convergent. As $M \rightarrow \infty$, we have

$$(1 - e^{-\tau M^2})^2 \rightarrow \Theta(\tau - M^{-2}), \quad (\text{B15})$$

where Θ is the Heaviside step function, reproducing Eq. (B7) with $\Lambda = M$.

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